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Separation in Nonlinear Time Models

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Little is known about the expressive completeness of connectives in temporal logic systems with a nonlinear time model. We introduce *separation*—a general tool for proving completeness in nonlinear time models. We then use the separation theorem to show expressive completeness of a finite set of connectives in various branching time models.

1. INTRODUCTION

Tense Logic is viewed as a Modal Logic where the relation between the universes of its model is an order relation. It is, however, interesting in itself since it offers a mechanism for reasoning about assertions that change with time. Most temporal systems proposed thus far treated time as a linear discrete system (as the natural numbers, for example). However there seems to be advantages in using nonlinear time models. Branching time is natural for dealing with parallel processing, infinite trees are natural models for nondeterminism. In this paper we address the issue of expressive completeness of connectives in nonlinear time models.

A basic decision in a logical system is the choice of connectives. Usually, an infinite number of connectives can be defined semantically. In the classical propositional calculus, for example, every truth table defines a connective. One would like to use a finite set that expresses all possible connectives. In the propositional calculus, $\neg$ and $\wedge$ are one such possible pair. In Tense Logic the addition of a time structure adds temporal connectives. Furthermore, the truth value of a formula may depend on the moments of time where the various components are evaluated and their relation to each other. Clearly, $\neg$ and $\wedge$ are not sufficient to express all possible such connectives. Kamp (1968) showed that over a linear, Dedekind-complete structure there is expressive completeness. The tense connectives that express all others are **Until** and **Since**. The time models in

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proposed temporal systems in the literature are linear and discrete, and the tense connectives are based on Kamp's expressively complete pair. Stavi (1979) showed that two more connectives are needed if the time model is linear and dense but has gaps in it (as do the rationals).

The choice of connectives for temporal systems introduced thus far in the literature seems to be ad hoc but as long as a set is expressively complete it is justifiable. As more nonlinear time models are considered the need for knowledge about expressive completeness in nonlinear time rises. Unfortunately, Gabbay (1981) showed that a general, partially ordered structure, yields an infinite number of independent connectives. All hope, however, is not lost since most users of Tense Logic have a specific time structure in mind. It is thus useful to classify the models where finite expressive completeness is possible. A tighter upper bound for models with expressive completeness was given by Amir (1984). It was proven that in an infinite tree with unbounded degree (for every natural k there is a node with more than k edges) finite expressive completeness is not possible.

The concept of completeness depends on a semantic definition of "truth table." Two such definitions are used in Tense Logic. A natural one and the most widely used, where the truth value of a formula depends on one "free" point in time—the "present," and a weaker definition introduced by Gabbay (1981) of "multidimensional" truth tables where many "free time points" define the value of a truth table. Some techniques for handling multidimensional tables and examples of nonlinear time models and their finite expressively complete sets of connectives were shown in (Amir and Gabbay, 1984).

In this paper the separation property is introduced. If a given set of connectives has this property then it is expressively complete *in one dimension.* We also use this property to show some finite one dimensional expressively complete sets of connectives in various branching time models. In a sequel we will use separation to show that an infinite bounded degree tree has finite expressive completeness. Separation is also general enough to be in other types of time models, such as branching future and past, for example.

In the next section the syntax and semantics of Tense Logic is reviewed. The concept of expressive completeness is defined in Section 3. The separation property is introduced in Section 4 and a theorem linking separation and expressive completeness is proven. In Section 5 the comb structures are defined and expressive completeness of $\mathscr{T}_2$ is shown. Section 6 inductively proves expressive completeness of $\mathscr{T}_n$ for all $n \geqslant 1$. The inductive proof in Section 6 covers the case of $n = 2$ so the reader interrested only in the theoretical results of this paper may skip Section 5. However the reader interrested in a rigorous example of using the separation theorem will find Section 5 quite useful.

2. Tense Logic—Syntax of wff and Semantics

Definitions 1. SYNTAX. Let atomic propositions be denoted by small letters such as $p, q, r, \ldots$, as in the propositional calculus, and denote the usual classical connectives by $\wedge$, $\vee$, $\neg$, $\rightarrow$. Let $\mathscr{C}$ be a set of *temporal connectives* denoted by combinations of capital letters such as U, S, P, F, NR, etc. Every temporal connective is associated with a natural number n of arguments, $n \geqslant 0$.

A *wff* in $\mathscr{C}$ is inductively defined, similarly to the propositional calculus, as:

- An atomic proposition is a *wff*.
- If $A \in \mathscr{C}$ is a n-argument temporal connective and if $\alpha_1, \ldots, \alpha_n$ are *wff*s then $A(\alpha_1, \ldots, \alpha_n)$ is a *wff*.
- If α, β are *wff*s then $\alpha \rightarrow \beta$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\neg\alpha$ are *wff*s.

A *Propositional Tense Logic* $\mathscr{S}$ with a set of temporal connectives $\mathscr{C}$ is a set of wffs in $\mathscr{C}$. The semantics of a tense logic is defined as follows:

SEMANTICS. Let $\langle \mathscr{T}, <, = \rangle$ be a partially ordered structure where $\mathscr{T}$ is nonempty, $<$ is a transitive relation on $\mathscr{T}$, and $=$ is the equality. Call $\langle \mathscr{T}, <, = \rangle$ a **flow of time** or a **time model**, call the elements of $\mathscr{T}$ **moments** of time $\mathscr{T}$, and say that a moment t is **earlier** (**later**) than a moment s if $t < s$ ($s < t$). Let $S(\mathscr{T})$ be the set of subsets of $\mathscr{T}$ and $\mathscr{P}$ the set of atomic propositions. A function $h: \mathscr{P} \rightarrow S(\mathscr{T})$ is called a **truth function** or **assignment**. An atomic proposition p is **true at a moment** t **under** h if $t \in h(p)$. Write it in symbols as $\|p\|_t^h = 1$; t is called the **evaluation point**.

If $t \notin h(p)$ then p is **false at a moment** t **under** h, or $\|p\|_t^h = 0$. The behavior of the connectives $\wedge$ and $\neg$ is determined by the required conditions:

$$\|\varphi \wedge \psi\|_t^h = 1 \quad \text{iff} \quad \|\varphi\|_t^h = \|\psi\|_t^h = 1,$$
$$\|\neg\varphi\|_t^h = 1 \quad \text{iff} \quad \|\varphi\|_t^h = 0.$$

Since $\vee$ and $\rightarrow$ can be defined using $\wedge$ and $\neg$, their truth values are defined accordingly.

While the truth values of atomic propositions and of the classical connectives are only dependent on the valuation point, other connectives could be more complicated. Some connectives may need **reference points** in time. For example, consider a truth connective $\#$ such that $\|\#(p, q)\|_{t,s}^h = 1$ ($\#(p, q)$ is true at evaluation point t with reference point s under assignment h) if proposition p is true at moment t and proposition q is true at moment s. One may choose to allow references to many evaluation points or take another approach and allow only one evaluation point (intuitively corresponding to the "present moment"). Traditionally, one evaluation point was allowed but Gabbay (1981) generalized it. We follow Gabbay in calling many evaluation point connectives *multidimensional* and single evaluation point connectives *one dimensional.* In systems having connectives with more than one evaluation point the truth value of all connectives is defined using the maximum number of points necessary. This is not a constraint and does not change the definition of connectives with less evaluation points since additional moments are ignored. For example, $\|p \vee q\|^h_{t,r_1,\ldots,r_{m-1}} = 1$ iff $\|p\|^h_{t,r_1,\ldots,r_{m-1}} = 1$ or $\|q\|^h_{t,r_1,\ldots,r_{m-1}} = 1$ iff $t \in h(p)$ or $t \in h(q)$. The notion of assignment is similarly extended to more than one moment with

$$h'(p) = \{(t, r_1, \ldots, r_{m-1})\colon t \in h(p)\}.$$

For convenience sake we will not distinguish henceforth between h' and h.

In the propositional calculus, connectives could be defined by the use of truth tables. We shall now define this concept in propositional tense logic.

DEFINITIONS 2. Let $\mathscr{L}$ be the full first-order predicate logic of $\langle \mathscr{T}, <, = \rangle$ using the symbols $<, =$ as constant predicates with the natural meaning, and let P_i, $i = 1, \ldots, n$ be symbols for m-place predicates over $\mathscr{T}$. Let $\psi(t, x_1, \ldots, x_{m-1}, <, =, P_1, \ldots, P_n)$ be a wff with the variables $t, x_1, \ldots, x_{m-1}$ free; ψ is called a n-place m-dimensional table over $\mathscr{T}$.

Let $\#$ be an n-place connective on tense logic formulas and $\psi_{\#}(t, x_1, \ldots, x_{m-1}, <, =, P_1, \ldots, P_n)$ be a n-place m-dimensional table; $\psi_{\#}$ can be used as a table defining $\#$ as follows:

$$\begin{aligned} &\|\#(\varphi_1, \ldots, \varphi_n)\|^h_{t,x_1,\ldots,x_{m-1}} = 1 \\ &\qquad \text{iff} \quad \langle \mathscr{T}, <, = \rangle \models \psi_{\#}(t, x_1, \ldots, x_{m-1}, <, =, h(\varphi_1), \ldots, h(\varphi_n)) \end{aligned}$$

where $h(\varphi) = \{(s, y_1, \ldots, y_{m-1})\colon \|\varphi\|^h_{s,y_1,\ldots,y_{m-1}} = 1\}$. Note that for atomic proposition p, $\|p\|^h_{t,y_1,\ldots,y_{m-1}} = 1$ iff $t \in h(p)$. In particular, a n-place 1-dimensional table over $\mathscr{T}$ is a table $\psi(t, <, =, P_1, \ldots, P_n)$, where $P_i \subseteq \mathscr{T}$, $i = 1, \ldots, n$.

Let us denote connectives by $\#$ and their respective tables by $\psi_{\#}$. If a set $C = \{(\#, \psi_{\#})\}$ is given, for any wff $A(p_1, \ldots, p_n)$, where p_i, $i = 1, \ldots, n$ are the propositions appearing in A, the truth value of $\|A(p_1, \ldots, p_n)\|^h_{t,\mathbf{x}}$ can be computed. Moreover, it can be easily shown by induction on the complexity of A that a table $\psi_A(t, \mathbf{x}, <, =, P_1, \ldots, P_n)$ exists in $\mathscr{L}$ such that for all $h, t, \mathbf{x}$,

$$\|A(p_1, \ldots, p_n)\|^h_{t,\mathbf{x}} = 1 \qquad \text{iff} \quad \langle \mathscr{T}, <, = \rangle \models \psi_A(t, \mathbf{x}, <, =, h(p_1), \ldots, h(p_n))$$

where $\mathbf{x}$ designates $x_1,\ldots, x_{m-1}$ and m is the maximum dimension of any $\psi_{\#}$. The question is if the converse is also true. Is there a finite set of connectives with which for any given table ψ_A a corresponding wff A can be built?

3. Expressive Completeness

Definitions 3. (a) Let there be given a m-dimensional tense system with connectives $\#_i$ and tables ψ_i. We say that the system is **expressively complete** (referred to in the literature also as **functionally complete**) in m-dimensions iff for any $\psi(t, x_1,\ldots, x_{m-1}, Q_1,\ldots, Q_k)$ of the language $\mathscr{L}$ there exists a wff $B(q_j, \#_i)$ built from the atoms $q_1,\ldots, q_k$ and the tense connectives such that for any; $h, t, x_1,\ldots, x_{m-1}$ we have

$$\langle \mathscr{T}, <, = \rangle \models \psi_B(t, \mathbf{x}, h(q_j)) \leftrightarrow \psi(t, \mathbf{x}, h(q_j)).$$

In other words for any ψ there exists a B such that $\psi = \psi_B$.

(b) The m-dimensional tense system is said to be **expressively complete in one dimension** iff for any $\psi(t, Q_j)$ of $\mathscr{L}$ with only t free and monadic Q_i there exists a B of the language such that

$$\langle \mathscr{T}, <, = \rangle \models \psi(t, Q_j) \leftrightarrow \psi_B(t, Q_j),$$

i.e., for any $\psi(t, Q_j)$ there exists a $B(q_j)$ such that for any h, t $\|B\|^h_t = 1$ iff $\mathscr{T} \models \psi(t, h(q_j))$.

For simpler terminology we use the following conventions:

(c) A flow of time $\langle \mathscr{T}, <, = \rangle$ is said to be **expressively complete** (or equivalently **functionally complete**) **in m-dimensions** iff there exists a finite set of m-dimensional connectives which is expressively complete in m-dimensions.

(d) $\langle \mathscr{T}, <, = \rangle$ is said to be **expressively (functionally) complete** if it is functionally complete in one dimension.

Example 4. Kamp (1968) showed that the two connections U, S with the classical connectives are expressively complete over the natural and real numbers (but not the rationals). $U(\alpha, \beta)$ intuitively means "β is true until α is true" and is defined as

$$\|U(\alpha, \beta)\|^h_t = 1 \qquad \text{iff} \quad \exists s > t\, ((\|\alpha\|^h_s = 1) \wedge \forall y\, ((t < y < s) \rightarrow (\|\beta\|^h_y = 1))).$$

$S(\alpha, \beta)$ intuitively means "β has been true since α was true" and is defined as

$$\|S(\alpha, \beta)\|^h_t = 1 \qquad \text{iff} \quad \exists s < t\, ((\|\alpha\|^h_s = 1) \wedge \forall y\, ((s < y < t) \rightarrow (\|\beta\|^h_y = 1))).$$

Note that we did not follow the exact definition of truth table. The full definition is more cumbersome. For the case of **Until** it would be

$$\psi_U(t, P, Q) = \exists s > t\ (P(s) \wedge \forall y\ ((t < y < s) \rightarrow Q(y)))$$

and then

$$\|U(\alpha, \beta)\|_t^h = 1 \qquad \text{iff} \quad \psi_U(t, h(\alpha), h(\beta)).$$

For convenience sake, truth tables will henceforth be written in the looser way, as used for U and S in the example, rather than stringently following the formal definition.

We note that the traditional meaning behind "completeness" in Tense Logic is not entirely identical to "completeness" in the classical propositional calculus. The difference is rooted in the definition of truth tables. A classical propositional calculus truth table gives all possible truth values to the propositions. In tense logic the truth value of a proposition is a set and all possible truth values would mean an uncountable number of values in any truth table as well as an uncountable number of truth tables in any infinite time model. Therefore, the definition of a truth table is traditionally restricted to the first-order propositional language over the time model. This restriction is not unnatural since the underlying tense logic of human speech makes the same assumption.

4. The Generalized Separation Property

We show a natural generalization to Gabbay's linear separation which enables proving one-dimensional finite expressive completeness in non-linear models.

Definition 5. Let $\mathscr{L}$ be the first order predicate language of $\langle \mathscr{T}, <, = \rangle$ using only the predicates $<$ and $=$. Let $\varphi_i(x, y)$, $i = 1,\dots, n$, be n given formulas in the language $\mathscr{L}$ of a structure $\mathscr{T}$. Assume that for every $t \in \mathscr{T}$ the following holds:

(i) The sets $T_{\varphi_i} = \{s \in \mathscr{T} : \varphi_i(s, t)\}$ $i = 1,\dots, n$ are mutually exclusive.

(ii) $\bigcup_{i=1}^n T_{\varphi_i} = \mathscr{T}$.

Then $\{\varphi_i\}_{i=1}^n$ is a *separating set*. Note that T_{φ_i} are dependent on t. For $s \neq t$ the sets S_{φ_i} are different from T_{φ_i}.

We say a formula α of tense logic $\mathscr{S}$ is *pure*-φ_i if for any $t \in \mathscr{T}$ and any two truth functions h and h' for which $\forall s \in T_{\varphi_i}$ and for all atomic propositions q, $s \in h(q)$ iff $s \in h'(q)$, we have

$$\|\alpha\|_t^h = \|\alpha\|_t^{h'}.$$

The tense logic $\mathscr{S}$ has the *generalized separation property* in time $\mathscr{T}$ if every formula α of $\mathscr{S}$ is equivalent to a boolean combination of formulas of the form β^{φ_i}, where $\{\varphi_i\}$ is a separating set, $<$ and $=$ can be expressed in $\mathscr{L}$ as boolean combinations of the φ_i and each β^{φ_i} is pure-φ_i.

The general separation property means that every wff in tense logic $\mathscr{S}$ with tense connectives $\mathscr{C}$ can be equivalently written in $\mathscr{S}$ as a boolean combination of subformulas each of which is local to some T_{φ_i}. Section 5 will rigorously demonstrate an example.

THE GENERALIZED SEPARATION THEOREM. *If a tense logic $\mathscr{S}$ has the generalized separation property for separating set $\{\varphi_i\}_{i=1}^n$ and if the following truth tables can be expressed by wffs in $\mathscr{S}$:*

$$\|C_{\varphi_i}A\|_t^h = 1 \leftrightarrow \exists s\ \varphi_i(s, t) \wedge \|A\|_s^h = 1$$

$$\|C_{\overline{\varphi_i}}A\|_t^h = 1 \leftrightarrow \exists s\ \varphi_i(t, s) \wedge \|A\|_s^h = 1$$

$$\|C_{\varphi_{ij}}A\|_t^h = 1 \leftrightarrow \exists s\ \varphi_i(s, t) \wedge \varphi_j(t, s) \wedge \|A\|_s^h = 1 \qquad i, j = 1, \ldots, n$$

then $\mathscr{S}$ with temporal connectives $\mathscr{C}$ is expressively complete.

Proof. Let $\varphi(t, <, =, Q_1, \ldots, Q_k)$ be a first-order formula with only $t \in \mathscr{T}$ free and $Q_i \subseteq \mathscr{T}$. We will prove by induction on the depth of quantifiers in φ that there is a wff formula A of $\mathscr{S}$ built of atomic formulas q_i such that for all $t \in \mathscr{T}$ and truth function h we have

$$\|A\|_t^h = 1 \leftrightarrow \langle \mathscr{T}, <, = \rangle \models \varphi(t, <, =, h(q_1), \ldots, h(q_k)).$$

First assume $\varphi = \exists s\ \psi(t, x)$, where ψ is a boolean combination of atomic formulas. These atomic formulas are of the form

$$t = x, \qquad t < x, \qquad x < t, \qquad Q_i(x), \qquad \text{and} \qquad Q_i(t) \qquad i = 1, \ldots, k.$$

$\mathscr{S}$ has the generalized separation property, so ψ is equivalent to a formula ψ' whose atoms are

$$\varphi_i(t, x), \varphi_i(x, t) \qquad i = 1, \ldots, n \qquad \text{and} \qquad Q_j(x), Q_j(t) \qquad j = 1, \ldots, k.$$

ψ' will now be rearranged in a manner that will yield the desired result. Assume ψ' is in disjunctive normal form. By pushing the quantifier into the formula, each disjunct can be handled separately. Every disjunct ψ'_g is a conjunction composed of $\varphi_{i_0}(x, t)$, $\varphi_{j_0}(t, x)$, $Q_j(x)$, and $Q_j(t)$, $j = 1, \ldots, k$. There is only one φ_{i_0} and one φ_{j_0} in a disjunct, since by definition $\neg\varphi_i = \bigvee_{j \neq i} \varphi_j$ and for $i \neq j$, $\varphi_i(x, y) \wedge \varphi_j(x, y)$ is a contradiction. Abbreviate "A or $\neg A$" by "$\pm A$." Since t is not bound by the quantifier, $\pm Q_i(t)$ can be extracted, leaving every disjunct with the form

$$\alpha \wedge \exists x\, (\varphi_i(x, t) \wedge \varphi_j(t, x) \wedge \beta(x)),$$

where α is a conjunction of $\pm Q_j(t)$ and β is a conjunction of $\pm Q_j(x)$, $j \in \{1, \ldots, k\}$.

Now take A_g to be $\alpha' \wedge C_{\varphi_{ij}}(\beta')$, where α' is constructed by replacing all appearances of Q_j in α by q_j and β' is constructed by replacing all appearances of $Q_j(x)$ in β by q_j.

By definition and the above construction

$$\|A_g\|_t^h = 1 \leftrightarrow \langle \mathscr{T}, <, = \rangle \models \psi'_g(t, h(q_j)).$$

For convenience we abbreviate $\psi(\ldots, h(q_1), \ldots, h(q_k), \ldots)$ as $\psi(\ldots, h(q_i), \ldots)$ throughout this proof. Now take $A = \bigvee_g A_g$ and the proof of the initial case is complete.

Induction hypothesis. For every $\psi(t, <, =, q_1, \ldots, q_k)$ which has no more than m nested quantifiers, there is a formula in $\mathscr{S}$ that has ψ as is its truth table. We have to prove that the same holds for a formula with $m+1$ quantifiers. Again it is sufficient to consider a formula $\exists x\, \psi(t, x)$ where the depth of quantifiers in ψ is not greater than m. Remember that ψ can be equivalently expressed using atomic formulas of the form:

$$\begin{aligned} Q_i(y), Q_i(t) \qquad & i = 1, \ldots, k, \\ \varphi_i(y, t) \qquad & i = 1, \ldots, n, \\ \varphi_i(y, y') \qquad & i = 1, \ldots, n, \end{aligned}$$

where y and y' are variables other than t; t is free in ψ so $\exists\psi(t, x)$ is equivalent to $\bigvee_g \alpha_g \wedge \exists x\, \psi_g(t, x)$, where α_g are conjunctions of $Q_i(t)$ and $\psi_g(t, x)$ are composed of all other atoms. The only atoms in ψ_g which include the variable t may be $\varphi_i(y, t)$ and $\varphi_i(t, y)$ for some $i = 1, \ldots, n$.

Our strategy will be to "get rid" of t in the formula ψ_g, then there will remain no more than m variables and depth of quantifiers no greater than m and the induction hypothesis will be used. Do this by substituting n new atomic relations $R^{\varphi_i}(y)$ for $\varphi_i(y, t)$ and $R^{\bar{\varphi}_i}(y)$ for $\varphi_i(t, y)$. The resulting formula $\psi_g^*(x, R^{\varphi_i}, Q_j)$ has depth of quantifiers not greater than m so by the induction hypothesis there is a formula $B_g(r_{\varphi_i}, q_j)$ in $\mathscr{S}$ such that

$$\|B_g\|_x^h = 1 \qquad \text{iff} \quad \langle \mathscr{T}, <, = \rangle \models \psi_g^*(x, h(r_{\varphi_i}), h(q_j))$$

and therefore

$$\left\| \bigvee_{i=1}^{n} C_{\varphi_i} B_g \right\|_t^h = 1 \qquad \text{iff} \quad \langle \mathscr{T}, <, = \rangle \models \bigvee_i \exists x(\varphi_i(x, t) \wedge \psi_g^*(x, h(r_{\varphi_j}), h(q_j))$$
$$\text{iff} \quad \langle \mathscr{T}, <, = \rangle \models \exists x\, \psi_g(t, x, h(q_j))$$

The remainder of the proof will be devoted to extracting the r_{φ_i} from the formula. The concept used will be similar to the linear case of the separation theorem (Gabbay, 1981). Without limiting generality it can be assumed that $\varphi_1(x, y)$ is $x = y$, since otherwise there is an i_0 for which $t \in T_{\varphi_{i_0}}$. Change φ_{i_0} to

$$\varphi'_{i_0} = \varphi_{i_0} \wedge (x \neq t)$$

and add $\varphi_0(x, y) = (x = y)$. The separation property remains since we can separate the original set then take the pure-φ_{i_0} formulas and extract the atoms from them.

The next step is to derive an equivalent formula to B_g with only one new atom r. Introduce a new atom r and replace r_{φ_1} in B_g by r. Now replace all appearances of r_{φ_i}, $i = 2, \ldots, n$ by $C_{\bar{\varphi}_i} r$. The resulting formulas A_g only have the atoms $r, q_1, \ldots, q_k$. For any truth function satisfying

(i) $\| r \wedge \neg C_{\varphi_2} r \wedge \cdots \wedge \neg C_{\varphi_n} r \|_t^h = 1$, the following holds:

$$\| r_{\varphi_1} \|_y^h = 1 \quad \text{iff} \quad y = t \text{ iff } R^{\varphi_1}(y) \text{ iff } \| r \|_y^h = 1;$$
$$\| r_{\varphi_i} \|_y^h = 1 \quad \text{iff} \quad \varphi_i(y, t) \text{ iff } R^{\varphi_i}(y) \text{ iff } \exists s\, \varphi_i(y, s) \wedge (s = t) \text{ iff } \| C_{\bar{\varphi}_i} r \|_y^h = 1;$$
$$\| r_{\bar{\varphi}_i} \|_y^h = 1 \quad \text{iff} \quad \varphi_i(t, y) \text{ iff } R^{\bar{\varphi}_i}(y) \text{ iff } \exists s\, \varphi_i(s, y) \wedge (s = t) \text{ iff } \| C_{\varphi_i} r \|_y^h = 1.$$

We conclude that for any h satisfying (i), for all x and t,

$$\| A_g \|_x^h = 1 \qquad \text{iff} \quad \langle \mathscr{T}, <, = \rangle \models \psi_g^*(x, h(r_{\varphi_i}), h(q_j)).$$

If we now take $A = \bigvee_g \alpha_g^* \wedge (A_g \vee C_{\varphi_2} A_g \vee \cdots \vee C_{\varphi_n} A_g)$, where α_g^* is the result of replacing q_j for Q_j in α_g we have a formula A as desired, but with one extra atom r. If the r is extracted from A, the proof of the induction step will be complete. The separation property will be used to eliminate the r.

Because of separation, A can be expressed as a boolean combination of pure-φ_i formulas $A_j^{\varphi_i}$, thus in disjunctive normal form

$$A = \bigvee_j (A_j^{\varphi_1} \wedge \cdots \wedge A_j^{\varphi_n}).$$

Therefore for any h that satisfies (i), $\|A\|_t^h = 1$ iff for at least one j, $\|A_j^{\varphi_i}\|_t^h = 1$ for all $i = 1, \dots, n$. Since $A_j^{\varphi_i}$ is composed of atoms and their negations, it may include the atom r. But because of property (i), r is true at point t and false at any other point, so if r is replaced by a taulology in $A_j^{\varphi_1}$ and by a contradiction in $A_j^{\varphi_i}$, $i = 2, \dots, n$, and if the resulting formulas are designated as $A_j^{\varphi_i^*}$ respectively we get

$$\|A_j^{\varphi_i}\|_t^h = \|A_j^{\varphi_i^*}\|_t^h.$$

Let $A^* = \bigvee_j (A_j^{\varphi_1^*} \wedge \cdots \wedge A_j^{\varphi_n^*})$. For any h satisfying condition (i) we get

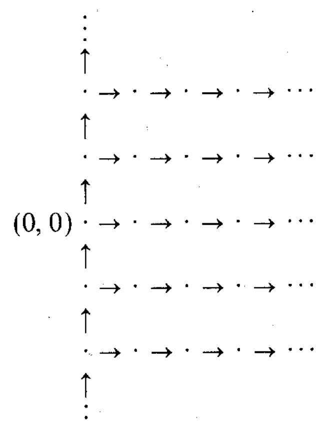
$$\|A^*\|_t^h = 1 \qquad \text{iff} \quad \langle \mathscr{T}, <, = \rangle \models \exists x\, \psi(t, x, h(q_1), \dots, h(q_n)).$$

However, condition (i) is a restriction on r which does not exist anymore, so the above is true for all h, and the proof of the induction step is completed. ▮

5. Expressive Completeness in the Comb Structure

The generalized separation theorem will be used to show expressive completeness in a family of "comb" structures $\{\mathscr{T}_n\}_{n=1}^{\infty}$ which are inductively defined as follows:

Definition 6. Define a partial order on the structure $N \times Z$ as follows: $(x_1, y_1) < (x_2, y_2)$ if $y_1 = y_2$ and $x_1 < x_2$ or $y_1 < y_2$ and $x_1 = 0$. A schematic of the order is



Definition 7. Designate the structure Z with the natural order as $\langle \mathscr{T}_1, <, = \rangle$. Designate the structure $N \times Z$ with the order relation defined in 6 as $\langle \mathscr{T}_2, <, = \rangle$. Let N with the natural order be $\langle \mathscr{T}'_1, <, = \rangle$ and

$N \times N$ with the order relation defined in 6 (contracted to the appropriate substructure) be $\langle \mathscr{T}'_2, <, = \rangle$. The structure $\langle \mathscr{T}_{n+1}, <, = \rangle$ is inductively defined as follows:

Let $\{t_i\}_{i=-\infty}^{\infty}$ be a linearly ordered sequence such that $t_i < t_{i+1}$ and $t_i \notin \mathscr{T}_n$ for all i. Let $\mathscr{T}'_{n,i}$ be the structure $\mathscr{T}'_n \cup \{t_i\}$ (for a fixed i) with $t_i < t$ $\forall t \in \mathscr{T}_n$ and the order between elements of $\mathscr{T}_n$ remaining unchanged. Now take

$$\mathscr{T}_{n+1} = \bigcup_{i=-\infty}^{\infty} \mathscr{T}'_{n,i}$$

and

$$\mathscr{T}'_{n+1} = \bigcup_{i=0}^{\infty} \mathscr{T}'_{n,i}$$

with the order defined as: The order of elements within $\mathscr{T}'_{n,i}$ remains unchanged:

If $x \in \mathscr{T}'_{n,i}$, $y \in \mathscr{T}'_{n,j}$ and $x \neq t_i$, $y \neq t_j$ then x and y are incomparable.

If $x = t_i$, $y = t_j$, and $i > j$ then $x > y$.

If $x \in \mathscr{T}'_{n,i}$, $y = t_j$ and $i > j$ then $x > y$.

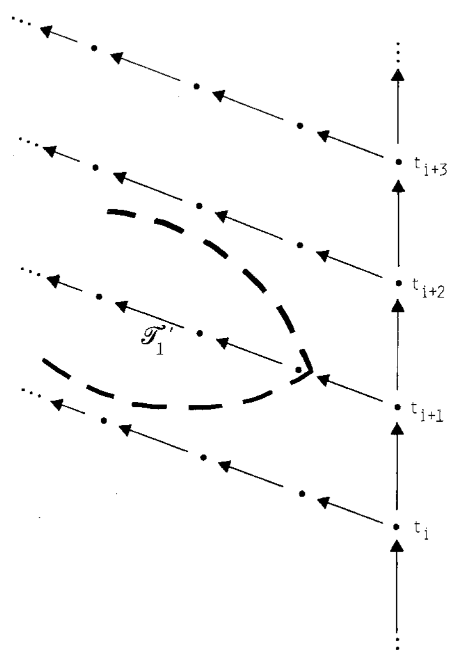


FIGURE 1

EXPLANATION. $\mathscr{T}_1$ is the linear discrete structure.

$\mathscr{T}_2$ is a *comb* made by "grafting" a copy of the naturals, $\mathscr{T}_1$ on every element of $\mathscr{T}_1$. Figure 1 is a schematic of $\mathscr{T}_2$.

$\mathscr{T}'_2$ is also a comb like $\mathscr{T}_2$ but it has a minimum point, i.e., pick a point on the "stem" of $\mathscr{T}_2$ and take only it and all the points above it. The result is $\mathscr{T}'_2$.

$\mathscr{T}_3$ is $\mathscr{T}_1$ with a copy of $\mathscr{T}'_2$ "grafted" on each of its points. See schematic in Fig. 2.

$\mathscr{T}'_3$ is derived from $\mathscr{T}_3$ by taking some point in the stem and all points above it. In general, $\mathscr{T}_{n+1}$ is $\mathscr{T}_1$ with a copy of $\mathscr{T}'_n$ "grafted" on each of its elements. See schematic in Fig. 3.

The following results have very tedious detailed proofs, however no new hard concept is involved. Once the connectives are defined the only difficulty is in proving separation. Expressive completeness then follows by the separation theorem. We prove separation in detail to show the method involved.

THEOREM 8. *The tense logic whose model is $\mathscr{T}_2$ is functionally complete.*

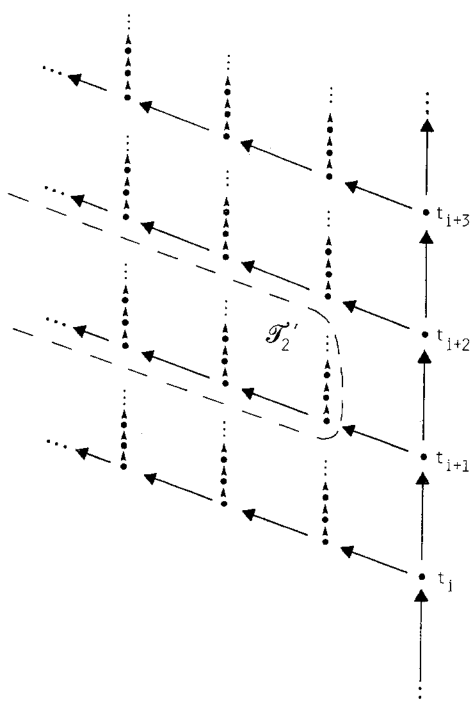


FIGURE 2

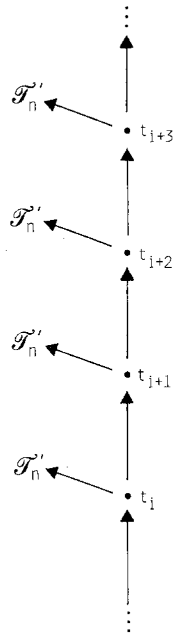


FIGURE 3

Proof. Use the generalized separation theorem. First, define the formulae φ_i, $i=0,\ldots,6$, to separate the structure at any given point. Extensive use shall be made of the following formula:

$$\tau(x)=\exists z\,\exists y(y>x)\wedge(z>x)\wedge\neg(y=z)\wedge\neg(y>z)\wedge\neg(y<z).$$

The meaning of $\tau(x)$ is "x is a branching point." Now take

$$\varphi_0(x,y)=(x=y)$$
$$\varphi_1(x,y)=\exists z((z>y)\wedge\tau(z))\wedge((x=z)\vee(x>z))$$
$$\varphi_2(x,y)=(x>y)\wedge\neg\exists z((x>z>y)\wedge\tau(z))\wedge\neg\tau(x)$$
$$\varphi_3(x,y)=(y>x)\wedge\neg\exists z((y>z>x)\wedge\tau(z))\wedge\neg\tau(x)$$
$$\begin{aligned}\varphi_4(x,y)=&[(y>x)\wedge\exists z((y\geqslant z>x)\wedge\tau(z))]\\&\vee[\exists z(x>z)\wedge\neg\tau(x)\wedge\neg\exists w((z<w<x)\wedge\tau(w))\\&\wedge(z<y)\wedge(\exists w((y>w>z)\wedge\tau(w))\vee(\tau(y)\wedge\tau(z)))]\end{aligned}$$
$$\begin{aligned}\varphi_5(x,y)=&\neg\tau(y)\wedge\exists z((y>z)\wedge\tau(z)\wedge\neg\exists w((y>w>z)\wedge\tau(w)\\&\wedge\exists w(\tau(w)\wedge(w>z)\wedge((x=w)\vee(x>w))))\end{aligned}$$
$$\varphi_6(x,y)=\tau(x)\wedge\neg\tau(y)\wedge\neg\exists z((y<z<x)\wedge\tau(z))\wedge(y>x).$$

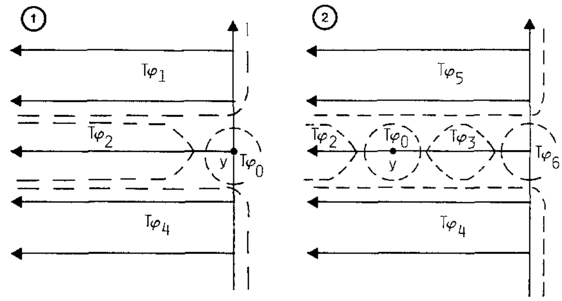


FIG. 4. (1) y on the spine; (2) y on a branch.

For a schematic of the sets $T\varphi_i$, $i=0,\ldots,6$, see Fig. 4. Note that when y is on the spine $T\varphi_3=T\varphi_5=T\varphi_6=\varnothing$ and when y is on a branch $T\varphi_1=\varnothing$. However, we will show that all conditions required by the cover theorem are still met:

A. $\bigcap_{i=0}^{6} T\varphi_i=\varnothing$. Note that intersecting $\varnothing\cap\varnothing=\varnothing$.

B. $\bigcup_{i=0}^{6} T\varphi_i=\mathscr{T}_2$. Again, adding the null set to a union does not alter it.

We also have that $\varphi_0(x, y)$ is $(x=y)$ and $\varphi_1(x, y)\vee\varphi_2(x, y)$ is $(x>y)$. Thus, all separation conditions are met.

We now need to define our connectives. The connectives U_r, S_r will be similar to U, S but defined only on the spine:

$$\|U_r(p, q)\|_t=1 \quad \text{iff} \quad \tau(t)\wedge\exists s((s>t)\wedge\tau(s)\wedge\|p\|_s=1)$$
$$\wedge\,(\forall u(s>u>t)\rightarrow\|q\|_u=1)$$
$$\|S_r(p, q)\|_t=1 \quad \text{iff} \quad \tau(t)\wedge\exists s((s<t)\wedge\tau(s)\wedge\|p\|_s=1)$$
$$\wedge\,(\forall u(s<u<t)\rightarrow\|q\|_u=1).$$

The connective U_b, S_b will be similar to U, S but defined only on a branch:

$$\|U_b(p, q)\|_t=1 \quad \text{iff} \quad \exists s[((s>t)\wedge\neg\tau(s)\wedge\forall u((s>u>t)\rightarrow\neg\tau(u))$$
$$\wedge\,\|p\|_s=1)\wedge(\forall u((s>u>t)\rightarrow\|q\|_u=1)]$$
$$\|S_b(p, q)\|_t=1 \quad \text{iff} \quad \exists s[((s<t)\wedge\neg\tau(s)\wedge\forall u((s<u<t)\rightarrow\neg\tau(u))$$
$$\wedge\,\|p\|_s=1)\wedge\forall u((s<u<t)\rightarrow\|q\|_u=1)].$$

One more connective is needed to enable passage from a branch to the spine:

$$\|L_b(p)\|_t=1 \quad \text{iff} \quad \neg\tau(t)\wedge\exists s[(s<t)\wedge\forall u((s<u<t)\rightarrow\neg\tau(u))$$
$$\wedge\,\tau(s)\wedge\|p\|_s=1].$$

It is necessary to prove that the above connectives have the generalized separation property. We first note that over $\mathscr{T}_2$

$$\vdash C_{\varphi 0}A \leftrightarrow A$$

$$\vdash C_{\varphi 1}A \leftrightarrow U_r(A, \text{tautology}) \vee U_r(U_b(A, \text{tautology}), \text{tautology})$$

$$\vdash C_{\varphi 2}A \leftrightarrow U_b(A, \text{tautology})$$

$$\vdash C_{\varphi 3}A \leftrightarrow S_b(A, \text{tautology})$$

$$\vdash C_{\varphi 4}A \leftrightarrow S_r(A, \text{tautology}) \vee S_r(U_b(A, \text{tautology}), \text{tautology}) \\ \vee L_b(S_r(A, \text{tautology})) \vee L_b(S_r(U_b(A, \text{tautology}), \text{tautology}))$$

$$\vdash C_{\varphi 5}A \leftrightarrow L_b(U_r(A, \text{tautology})) \vee L_b(U_r(U_b(A, \text{tautology}), \text{tautology}))$$

$$\vdash C_{\varphi 6}A \leftrightarrow L_b(A);$$

also,

$$C_{\bar\varphi 0}A = A$$
$$C_{\bar\varphi 1}A = S_r(A, \text{taut}) \vee L_b(S_r(A, \text{taut}))$$
$$C_{\bar\varphi 2}A = S_b(A, \text{taut}) \vee L_b(A)$$
$$C_{\bar\varphi 3}A = U_b(A, \text{taut}) \wedge L_b(\text{taut})$$
$$C_{\bar\varphi 4}A = U_r(A, \text{taut}) \vee U_r(U_b(A, \text{taut}), \text{taut}) \vee L_b(U_r(A, \text{taut})) \\ \vee L_b(U_r(U_b(A, \text{taut}), \text{taut}))$$
$$C_{\bar\varphi 5}A = S_r(U_b(A, \text{taut}), \text{taut}) \vee L_b(S_r(U_b(A, \text{taut}), \text{taut}))$$
$$C_{\bar\varphi 6}A = U_b(A, \text{taut}) \wedge \neg L_b(\text{taut});$$
$$C_{\varphi 00}A = A$$
$$C_{\varphi 14}A = C_{\varphi 1}A$$
$$C_{\varphi 26}A = C_{\varphi 2}A \wedge \neg L_b(\text{taut})$$
$$C_{\varphi 23}A = C_{\varphi 2}A \wedge L_b(\text{taut})$$
$$C_{\varphi 32}A = C_{\varphi 3}A$$
$$C_{\varphi 41}A = S_r(A, \text{taut}) \vee L_b(S_r(A, \text{taut}))$$
$$C_{\varphi 45}A = S_r(U_b(A, \text{taut}), \text{taut}) \vee L_b(S_r(U_b(A, \text{taut}), \text{taut}))$$

$$C_{\varphi_{54}}A = C_{\varphi_5}A$$
$$C_{\varphi_{62}}A = C_{\varphi_6}A.$$

All the rest of the $C_{\varphi_{ij}}$ are contradictions.

To show separation of the connectives we make use of the following facts:

1. Any formula composed of nested U_r and S_r can be separated to a boolean combination of pure-φ_1, pure-φ_0, and pure-φ_4 formulae as shown for the linear case by Gabbay (1981).

2. Any formula composed of nested U_b and S_b can likewise be separated to a boolean combination of pure-φ_2, pure-φ_0, and pure-φ_3 formulae.

3. As in the case of U, S, we have

$$\vdash U_b(A \vee B, C) \leftrightarrow U_b(A, C) \vee U_b(B, C)$$
$$\vdash U_b(A, B \wedge C) \leftrightarrow U_b(A, B) \wedge U_b(A, C).$$

The above formulae are true if we exchange U_b for S_b, U_r for S_r. Writing **cont** for contradiction and **taut** for tautology, by definition we have

4. $\vdash U_b(a \wedge U_r(b, c), q) \leftrightarrow U_b(\text{cont}, q)$
5. $\vdash U_b(a \wedge S_r(b, c), q) \leftrightarrow U_b(\text{cont}, q)$
6. $\vdash U_b(a \wedge \neg U_r(b, c), q) \leftrightarrow U_b(a, q)$
7. $\vdash U_b(a \wedge \neg S_r(b, c), q) \leftrightarrow U_b(a, q)$
8. $\vdash U_b(a, q \vee U_r(b, c)) \leftrightarrow U_b(a, q)$
9. $\vdash U_b(a, q \vee S_r(b, c)) \leftrightarrow U_b(a, q)$
10. $\vdash U_b(a, q \vee \neg U_r(b, c)) \leftrightarrow U_b(a, \text{taut})$
11. $\vdash U_b(a, q \vee \neg S_r(b, c)) \leftrightarrow U_b(a, \text{taut})$.

The same holds true if U_b is exchanged for S_b. For cases of nested L_b in U_b or S_b we have

12. $\vdash U_b(a \wedge L_b(c), q) \leftrightarrow (U_b(a, q) \wedge L_b(c)) \vee ((\neg L_b(\text{taut})) \wedge c \wedge U_b(a, q))$
13. $\vdash U_b(a, q \vee L_b(c)) \leftrightarrow U_b(a, q) \vee (U_b(a, \text{taut}) \wedge L_b(c)) \vee (U_b(a, \text{taut}) \wedge c \wedge \neg L_b(\text{taut}))$
14. $\vdash \neg L_b(c) \leftrightarrow \neg L_b(\text{taut}) \vee L_b(\neg c)$, therefore,
15. $\vdash U_b(a \wedge \neg L_b(c), q) \leftrightarrow (U_b(a, q) \wedge L_b(\neg c)) \vee ((\neg L_b(\text{taut})) \wedge \neg c \wedge U_b(a, q))$
16. $\vdash U_b(a, q \vee \neg L_b(\text{taut})) \leftrightarrow U_b(a, q)$
17. $\vdash S_b(a \wedge L_b(c), q) \leftrightarrow (S_b(a, q) \wedge L_b(c))$

18. $\vdash S_b(a, q \vee L_b(c)) \leftrightarrow S_b(a, q) \vee (S_b(a, \text{taut}) \wedge L_b(c))$

19. $\vdash S_b(a \wedge \neg L_b(c), q) \leftrightarrow S_b(a, q) \wedge L_b(\neg c)$

20. $\vdash S_b(a, q \vee \neg L_b(c)) \leftrightarrow S_b(a, q) \vee (S_b(a, \text{taut}) \wedge L_b(\neg c))$.

21. If S_b or L_b are nested in U_r or S_r, put cont instead of the S_b or L_b.

22. If $\neg S_b$ or $\neg L_b$ are nested in U_r or S_r, put taut instead of the $\neg S_b$ or $\neg L_b$.

23. $U_r(a \wedge \pm U_b(b, c), q)$ is pure-φ_1.

24. $U_r(a, q \vee \pm U_b(b, c))$ is pure-φ_1.

25. $S_r(a \wedge \pm U_b(b, c), q)$ is pure-φ_4.

26. $S_r(a, q \vee \pm U_b(b, c))$ is pure-φ_4.

27. $\vdash L_b(p \wedge q) \leftrightarrow L_b(p) \wedge L_b(q)$.

28. $\vdash L_b(p \vee q) \leftrightarrow L_b(p) \vee L_b(q)$.

29. $L_b(\pm U_r(p, q))$ is pure-φ_5.

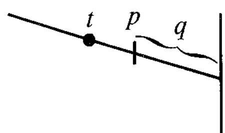
30. $L_b(\pm S_r(p, q))$ is pure-φ_4.

31. $\vdash L_b(S_b(p, q)) \leftrightarrow L_b(\text{cont})$.

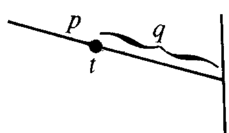
32. $\vdash L_b(\neg S_b(p, q)) \leftrightarrow L_b(\text{taut})$.

33. The last remaining case is $L_b(U_b(p, q))$. This can be divided to one of three cases:

I. $S_b(p \wedge S_b(\neg S_b(\text{taut}, \text{taut}) \wedge q, q), \text{taut})$. Note that $\neg S_b(\text{taut}, \text{taut})$ is only true in the smallest point on a branch.

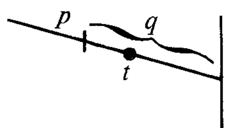


II. $p \wedge S_b(\neg S_b(\text{taut}, \text{taut}) \wedge q, q)$.



III. $U_b(p, q) \wedge q \wedge S_b(\neg S_b(\text{taut}, \text{taut}) \wedge q, q)$.

$$\vdash L_b(U_b(p, q)) \leftrightarrow \text{I} \vee \text{II} \vee \text{III}.$$



34. For the case of $L_b(\neg U_b(p, q))$, note that

$$\vdash L_b(\neg A) \leftrightarrow \neg L_b(A) \wedge L_b(\text{taut}).$$

35. $\vdash L_b(L_b(p)) \leftrightarrow \text{cont}.$

We now have sufficient tools to prove separation.

PROPOSITION 9. *Let α be a wff in a propositional calculus with connectives U_r, S_r, U_b, S_b, and L_b in addition to the classical connectives. Then α can be separated to a boolean combination of pure-φ_i formulae, $i = 0,\ldots, 6$. Furthermore, every pure-φ_2 formula will be boolean combination of nested U_b's only, every pure-φ_1 formula will be boolean combinations of nested U_r's and (possibly) U_b's with no U_r in the scope of U_b, and every pure-φ_4 formula will be a boolean combination of nested S_r's and (possibly) U_b's with no S_r in the scope of U_b, or such a combination nested in L_b.*

Proof. By induction on the number of nested non-classical connectives in α.

Basic Case. $\alpha = C(\bar{\beta})$, where $\bar{\beta}$ is composed of boolean combinations of atoms and C is non-classical, then by definition:

If $C = U_r$ then α is pure-φ_1,
if $C = S_r$ then α is pure-φ_4,
if $C = U_b$ then α is pure-φ_2,
if $C = S_b$ then α is pure-φ_3,
if $C = L_b$ then α is pure-φ_6.

For a classical C, α is pure-φ_0.

Induction step. Assume that any formula with n nested non-classical connectives can be separated, and $\alpha = C(\bar{\beta})$, where $\bar{\beta}$ is composed of formulae with n nested non-classical connectives. By the induction hypothesis, $\alpha = C(\bar{\beta}')$, where $\bar{\beta}'$ is composed of boolean combinations of pure-φ_i formulae.

The following cases should be analyzed:

A. $\alpha = U_r(\beta_1, \beta_2)$. Any pure-$\varphi_5$ or pure-φ_6 formula of β_1 is nested in L_b or $\neg L_b$ and by cases 21, 22 can be replaced with cont or taut, respectively. Any pure-φ_3 formula of β_i is nested in S_b or $\neg S_b$ and again by cases 21, 22 can be replaced with cont or taut, respectively.

Pure-φ_0 cases are boolean combinations of atoms.

Pure-φ_2 cases are nested U_b or $\neg U_b$, consider every such formula as a single atom.

Pure-φ_1 cases are nested in U_r and may include only the connectives U_r, U_b.

Pure-φ_4 cases are nested in S_r and may include only the connectives S_r, U_b.

Again consider every formula nested in U_b as a single atom. By the induction hypothesis there was no S_r or U_r in the U_b scope. Call the formula derived from α by the above changes α'. Because of case 1, α' can be separated, since it contains only U_r and S_r. The separated formula will be separated into pure-φ_1, pure-φ_0, and pure-φ_4 formulae. Now replace every U_b "atom" with the original formula in α.

By definition every pure-φ_1 formula will remain pure-φ_1 after exchanging atoms in it with a U_b formula, and a pure-φ_4 formula will remain pure-φ_4 after such an exchange. A pure-φ_0 formula will become pure-φ_2.

All pure-φ_2 formulae will have only nested U_b's.

All pure-φ_1 formulae will have nested U_b's in U_r's (with no U_r nested in U_b).

All pure-φ_4 formulae will have nested U_b's in S_r's (with no S_r nested in U_b).

B. $\alpha = S_r(\beta_1, \beta_2)$. This case is entirely similar to case A.

C. $\alpha = U_b(\beta_1, \beta_2)$. Any pure-$\varphi_1$ formula of β_i is nested in U_r or $\neg U_r$ and by cases 4, 6, 8, and 10 could be replaced by cont or taut, respectively.

Any pure-φ_0 formula is a combination of atoms.

Any pure-φ_6 or pure-φ_5 formula is nested in L_b, thus by cases 12–20 can be "moved out" of α, remaining pure-φ_6 or pure-φ_5, respectively.

Any pure-φ_4 formula is either nested in S_r or $\neg S_r$, in which cases by 5, 7, 9, 11 it could be replaced by cont or taut, respectively, or it is nested in L_b and by cases 12–20 can be "moved out" of α, remaining pure-φ_4, with no S_r's nested in U_b's. Thus we get an α' which is a boolean combination of pure-φ_4, pure-φ_5, and pure-φ_6 formulae and $U_b(\beta'_1, \beta'_2)$, where β'_i is a boolean combination of atoms, U_b's and S_b's. Because of Gabbay's results (1981), this can be separated into pure-φ_0, pure-φ_2 (involving only U_b's and pure-φ_3 (involving only S_b's) formulae.

D. $\alpha = S_b(\beta_1, \beta_2)$. This case is entirely similar to case C.

E. $\alpha = L_b(\beta)$. Because of 27, 28 it is sufficient to consider the cases where β itself is pure-φ_i.

For β pure-φ_0, α is pure-φ_6.

For β pure-φ_1, α is pure-φ_5, by case 29.

For β pure-φ_3, α is pure-φ_6, by cases 30, 31.

For β pure-φ_4, if it is nested in L_b or $\neg L_b$, α is pure-φ_6 by case 35, if it is nested in S_r, α is pure-φ_4 by case 30, with no S_r nested in U_b.

For β pure-φ_5 or pure-φ_6, β is a formula nested in L_b or $\neg L_b$, thus α is L_b(cont) or L_b(taut), respectively, pure-φ_6.

For β pure-φ_2, there are only nested U_b's in β. By case 33 the L_b can be done away with, leaving a formula having only nested U_b's, S_b's, and atoms and by case 2 this can be separated to pure-φ_0, pure-φ_2, and pure-φ_3 cases. Since this is the linear case, the pure-φ_2 formula has nested U_b's only, by Gabbay's result (1981). This concludes the induction step.

Thus any formula can be separated and by the generalized separation theorem, $\mathcal{T}_2$ is functionally complete. ▌

6. Functional Completeness in $\mathcal{T}_n$, $n \geqslant 1$

We have defined in Section 5 all structures in the family $\{\mathcal{T}_n\}_{n=1}^{\infty}$. Kamp (1968) tells us that $\mathcal{T}_1$ is functionally complete, and Theorem 8 tells us that $\mathcal{T}_2$ is functionally complete. We shall show inductively that $\mathcal{T}_n$ is functionally complete for all n. But first we need the following lemma:

The Weak Cover Lemma. *Let M_1, M_2 be expressively complete separable structures, $P_1^1,\ldots, P_n^1$ the connectives of M_1 by which all other connectives can be expressed, $P_1^2,\ldots, P_m^2$ the connectives of M_2 by which all other connectives can be expressed. Assume M_2 satisfies the following condition:*

There is an element $x \in M_2$ such that for every $y \in M_2$, $y \geqslant x$, call $x = \min M_2$.

Define a structure M_{12}: On every $x \in M_1$ graft a copy of M_2 such that in addition to the order relation in M_1, $x < \min M_2$. M_{12} is functionally complete if the following conditions are satisfied:

A. *There exists a set of connectives $P_1^{12},\ldots, P_n^{12}$ such that*

$$\vdash_{M_1} P_i^{12} \leftrightarrow P_i^1$$

and such that $\|P_i^{12}(\boldsymbol{\alpha})\|_t = 0$ if t is an element of one of the grafted copies of M_2.

B. *There exists a set of connectives $P_1^{21},\ldots, P_m^{21}$ such that*

$$\vdash_{M_2} P_i^{21} \leftrightarrow P_i^2$$

and such that $\|P_i^{12}(\boldsymbol{\alpha})\|_t = 0$ if t is an element of M_1.

C. *There exists a connective L^{21} such that*

$$\|L^{21}(\alpha)\|_t = 1 \quad \text{iff} \quad (t \in M_2) \wedge \exists s((s \in M_1) \wedge (t > s) \wedge \forall u((t > u > s) \rightarrow u \in M_2) \wedge \|\alpha\|_s = 1.$$

D. *There exists a connective G^{12} such that*

$$\|G^{12}(\alpha)\|_t = 1 \quad \text{iff} \quad (t \in M_1) \wedge \exists s((s \in M_2) \wedge (s > t) \wedge \forall u((s > u > t) \to u \in M_2) \wedge \|\alpha\|_s = 1.$$

Proof. Make use of the generalized separation theorem. Let $\varphi_0^1, \ldots, \varphi_k^1$ be the separating formulae of M_1, $\varphi_0^1(x, y) = (x = y)$, $\varphi_0^2, \ldots, \varphi_l^2$ be the separating formulae of M_2, $\varphi_0^2(x, y) = (x = y)$. The separating formulae of M_{12} will be

$$\varphi_0^{12}, \ldots, \varphi_k^{12}, \varphi_{k+1}^{12}$$

where $\varphi_0^{12}(x, y) = (x = y)$. $T\varphi_0^{12}$ defines only the element itself, where for $i = 1, \ldots, k$,

$$\varphi_i^{12}(x, y) = [(x \in M_1) \wedge (y \in M_1) \wedge \varphi_i^1(x, y)] \vee [(y \in M_1) \wedge \exists z((z \in M_1) \wedge (x > z) \wedge (x \in M_2)) \wedge \forall u((x > u > z) \to (u \in M_2)) \wedge \varphi_i^1(z, y)].$$

$T\varphi_i^{12}$ defines $T\varphi_i^1$ with the addition of all M_2 grafted on elements of $T\varphi_i^1$. This only for elements of M_1. For all other elements $T\varphi_i^{12} = \varnothing$:

$$\varphi_{k+1}^{12}(x, y) = (y \in M_1) \wedge (x > y) \wedge (x \in M_2) \wedge \forall u((x > u > y) \to (u \in M_2)).$$

$T\varphi_{k+1}^{12}$ is the M_2 grafted on t if $t \in M_1$, otherwise $T\varphi_{k+1}^{12} = \varnothing$.

Other separating formulae of M_{12} are

$$\varphi_1^{21}, \ldots, \varphi_l^{21}, \varphi_{l+1}^{21}, \ldots, \varphi_{l+k+1}^{21}$$

where for $i = 1, \ldots, l$,

$$\varphi_i^{21}(x, y) = (x \in M_2) \wedge (y \in M_2) \wedge \forall u((x < u < y) \to (u \in M_2)) \wedge \forall u((y < u < x) \to (u \in M_2)) \wedge \varphi_i^2(x, y).$$

$T\varphi_i^{21}$ defines $T\varphi_i^2$ for elements on an M_2, otherwise $T\varphi_i^{21} = \varnothing$,

$$\varphi_{l+1}^{21}(x, y) = (x < \min M_2) \wedge (y > x) \wedge (y \in M_2) \wedge \forall u((y > u > x) \to (u \in M_2)).$$

$T\varphi_{l+1}^{21}$ defines the root of the copy of M_2 on which t resides; the root is in M_1 by definition. For $t \in M_1$, $T\varphi_{l+1}^{21} = \varnothing$. For $i = 1, \ldots, k$,

$$\varphi_{l+1+i}^{21}(x, y) = \exists z(\varphi_{l+1}^{21}(z, y)) \wedge \varphi_i^{12}(x, z).$$

$T\varphi^{21}_{l+1+i}$ is $T\varphi^{12}_i$ for the root of the M_2 on which t resides. If $t \in M_1$, $T\varphi^{21}_{l+1+i} = \varnothing$.

We have to show that the new φ^{12}_i, φ^{21}_j, $i = 0,..., k+1$, $j = 1,..., l+k+1$, satisfy the conditions of separation for M_{12}:

A. If $t \in M_1$, all $T\varphi^{21}_i = \varnothing$, and since by definition every two copies of M_2 are mutually exclusive and $T\varphi^1_j$ are mutually exclusive, then $T\varphi^{12}_j$ which are created from $T\varphi^1_j$ by adding to them their grafted M_2, must also be mutually exclusive.

If $t \in M_2$, all $T\varphi^{12}_i = \varnothing$. $T\varphi^{21}_j$ are exactly $T\varphi^2_j$ on the M_2, where t resides for $j = 0,..., l$ and $T\varphi^{21}_{l+1}$ is the root. The rest of $T\varphi^{21}_j$ are like the case of $t \in M_1$, excluding the copy of M_2 on which t resides.

B. If $t \in M_1$, $\bigcup T\varphi^1_j = M_1$, but in $T\varphi^{12}_j$ are added all the grafted copies of M_2 so $\bigcup T\varphi^{12}_j = M_{12}$. If $t \in M_2$, this copy is covered by $\bigcup_{j=0}^{l+1} T\varphi^{21}_j$. The rest of M_{12} is covered by $\bigcup_{j=l+2}^{l+k+1} T\varphi^{21}_j$.

We now need to prove that $C\varphi^{12}_i$, $C\varphi^{21}_j$, $i = 0,..., k$, $j = 1,..., l+k+1$, can be expressed as boolean combinations of the connectives P^{12}_i, P^{21}_j, $i = 1,..., n$, $j = 1,..., m$:

$$\|C\varphi^{12}_0(\alpha)\|_t = 1 \quad \text{iff} \quad \|\alpha\|_t = 1.$$

Let $B^1_{i_0}$ be a boolean combination of P^1_i's such that

$$\|C\varphi^1_{i_0}(\alpha)\|_t = 1 \quad \text{iff} \quad \|B^1_{i_0}(\alpha)\|_t = 1.$$

Let $B^{12}_{i_0}$ be the formula derived from $B^1_{i_0}$ by exchanging all appearances of P^1_i by P^{12}_i in $B^1_{i_0}$. Then for $i = 1,..., k$, we have

$$\|C\varphi^{12}_i(\alpha)\|_t = 1 \quad \text{iff} \quad \|B^{12}_i(\alpha \vee G^{12}(\alpha))\|_t = 1$$

$$\|C\varphi^{12}_{k+1}(\alpha)\|_t = 1 \quad \text{iff} \quad \|G^{12}(\alpha)\|_t = 1.$$

Let $B^2_{i_0}$ be a boolean combination of P^2_i's such that

$$\|C\varphi^2_{i_0}(\alpha)\|_t = 1 \quad \text{iff} \quad \|B^2_{i_0}(\alpha)\|_t = 1.$$

Let $B^{21}_{i_0}$ be the formula derived from $B^2_{i_0}$ by exchanging all appearances of P^2_i by P^{21}_i in $B^2_{i_0}$. Then for $i = 1,..., l$, we have $\|C\varphi^{21}_i(\alpha)\|_t = 1$ iff $\|B^{21}_i(\alpha)\|_t = 1$:

$$\|C\varphi^{21}_{l+1}(\alpha)\|_t = 1 \quad \text{iff} \quad \|L(\alpha)\|_t = 1$$

$$\|C\varphi^{21}_{l+i+1}(\alpha)\|_t = 1 \quad \text{iff} \quad \|L(B^{12}_i(\alpha))\|_t = 1 \text{ for } i = 1,..., k.$$

We further have, $C_{\bar{\varphi}^{12}_0} = A$,

$$i = 1,..., k, \qquad C_{\bar{\varphi}^{12}_i} A = C'_{\bar{\varphi}^1_i}(A) \vee L^{21}(C'_{\bar{\varphi}^1_i}(A)),$$

where $C'_{\bar{\varphi}_i^1}$ is $C_{\bar{\varphi}_i^1}$ with the exchange of all P_j^1 by P_j^{12} respectively $(j \in \{1,\ldots, n\})$;

$$C_{\bar{\varphi}_{k+1}^{12}} A = L^{21}(A),$$
$$i = 1,\ldots, l, \qquad C_{\bar{\varphi}_i^{21}} A = C'_{\bar{\varphi}_i^2} A,$$

where $C'_{\bar{\varphi}_i^2}$ is $C_{\bar{\varphi}_i^2}$ with the exchange of all P_j^2 by P_j^{21} respectively $(j \in \{1,\ldots, m\})$;

$$C_{\bar{\varphi}_{l+1}^{21}} A = G^{12}(A),$$
$$i = 1,\ldots, k, \qquad C_{\bar{\varphi}_{l+1+i}^{21}} A = C'_{\bar{\varphi}_i^1}(G(A)) \vee L(C'_{\bar{\varphi}_i^1}(G(A))$$
$$i, j = 1,\ldots, k, \qquad C_{\varphi_{ij}^{12}} A = C'_{\varphi_{ij}^1} A,$$

where $C'_{\varphi_{ij}^1}$ is $C_{\varphi_{ij}^1}$ with the exchange of all P_d^1 by P_d^{12} respectively $(d \in \{1,\ldots, n\})$;

$$i, j = 1,\ldots, l, \qquad C_{\varphi_{ij}^{21}} A = C'_{\varphi_{ij}^2} A,$$

where $C'_{\varphi_{ij}^2} A$ is $C_{\varphi_{ij}^2} A$ with the exchange of all P_d^2 by P_d^{12} respectively $(d \in \{1,\ldots, m\})$;

$$i, j = 1,\ldots, k, \qquad C_{\varphi^{21}_{(l+1+i)(l+1+j)}} A = L(C'_{\varphi_{ij}^1}(G(A)))$$
$$C_{\varphi^{1221}_{(k+1)(l+1)}} A = C_{\varphi^{12}_{k+1}} A$$
$$C_{\varphi^{2112}_{(l+1)(k+1)}} A = C_{\varphi^{21}_{l+1}} A;$$
$$i, j = 1,\ldots, k, \qquad C_{\varphi^{1221}_{i(l+1+j)}} A = C'_{\varphi_{ij}^1}(G(A));$$
$$i, j = 1,\ldots, k, \qquad C_{\varphi^{2112}_{(l+1+i)j}} A = L(C'_{\varphi_{ij}^1} A).$$

All the rest of the $C_{\varphi_{ij}}$ are contradictions.

We proceed now in showing that every truth table defines a boolean combination of the connectives $P_1^{12},\ldots, P_n^{12}$, $P_1^{21},\ldots, P_m^{21}$, L^{21}, G^{12}. As in Theorem 8, start by considering all nested cases. P_i^{12} nested in P_j^{12} can be separated as P_i^1 nested in P_j^1, for all $i, j \in \{1,\ldots, n\}$. The same holds true for P_i^{21} nested in P_j^{21} as in P_i^2 and P_j^2. P_i^{12} nested in P_j^{21} can be exchanged with a contradiction. $\neg P_i^{12}$ nested in P_j^{21} can be exchanged with a tautology. This is a direct result of the domains of the connectives. The same is true for nested appearances of P_i^{21} and $\neg P_i^{21}$ in P_j^{12}.

The cases of L^{21} and G^{12} should now be considered. For convenience sake, write L and G for L^{21} and G^{12}, respectively. It can be easily seen that we have for L,

$$\vdash L(\alpha \wedge \beta) \leftrightarrow L(\alpha) \wedge L(\beta)$$

$$\vdash L(\alpha \vee \beta) \leftrightarrow L(\alpha) \vee L(\beta)$$

$$\vdash \neg L(\alpha) \leftrightarrow L(\neg\alpha) \vee \neg L(\text{tautology}).$$

It is sufficient to consider the cases of a single connective nested in L. $L(L(\alpha))$ is always a contradiction. $\|L(G(\alpha))\|_t = 1$ iff $\exists s\ \|\alpha\|_s = 1$ and both s and t reside on the same copy of M_2. Let $B^{21}_{i_0}$ be the boolean combination of P^{21}_j such that

$$\vdash C^{21}_{\varphi_{i_0}}(\alpha) \leftrightarrow B^{21}_{i_0}(\alpha).$$

Then $\vdash L(G(\alpha)) \leftrightarrow L(\text{tautology}) \wedge (\bigvee_{i=1}^{l} B^{2i}_i(\alpha) \vee \alpha)$.

As was previously mentioned, every formula having only the connectives P^{21}_j, $j = 1, \ldots, m$, in addition to the classical ones, can be separated. $L(\pm P^{12}_i)$ is pure-$\varphi^{21}_{i_0+l+1}$, where P^{12}_i is pure-$\varphi^{12}_{i_0}$:

$$\vdash L(P^{21}_i) \leftrightarrow L(\text{contradiction})$$

$$\vdash L(\neg P^{21}_i) \leftrightarrow L(\text{tautology}).$$

Every L nested in P^{12}_i can be exchanged with a contradiction:

$$\vdash P^{21}_i(\bar{\alpha}_1, \alpha_2 \wedge \pm L(\alpha_3), \bar{\alpha}_4) \leftrightarrow P^{21}_i(\bar{\alpha}_1, \alpha_2, \bar{\alpha}_4) \wedge \pm L(\alpha_3)$$

$$\vdash P^{21}_i(\bar{\alpha}_1, \alpha_2 \vee \pm L(\alpha_3), \bar{\alpha}_4) \leftrightarrow P^{21}_i(\bar{\alpha}_1, \alpha_2, \bar{\alpha}_4) \vee \pm L(\alpha_3)$$

$$\vdash G(\pm L(\alpha) \wedge \beta) \leftrightarrow G(\text{tautology}) \wedge G(\beta) \wedge \pm\alpha.$$

The case of G is not quite as fortunate as that of L but we still have

$$\vdash G(\alpha \vee \beta) \leftrightarrow G(\alpha) \vee G(\beta).$$

Thus we have to consider conjunctions nested in G. G nested in G can be exchanged with a contradiction. P^{12}_i nested in G can be exchanged with a contradiction. $G(\alpha)$, where α is a conjunction of P^{21}_i and atoms is pure-φ^{12}_{k+1}. G nested in P^{21}_i can be exchanged with a contradiction. If G is nested in P^{12}_i, where P^{12}_i is pure-$\varphi^{12}_{i_0}$, the formula remains pure-$\varphi^{12}_{i_0}$.

All conditions required by the generalized separation theorem will be met, as soon as separation is shown, with the aid of the following proposition.

PROPOSITION 10. *Let α be a wff in a propositional calculus with connectives P^{12}_i, $i = 1, \ldots, n$, P^{21}_i, $i = 1, \ldots, m$, L^{21}, and G^{12}, in addition to the classical connectives. Then α can be separated to a boolean combination of pure-φ^{12}_i, $i = 0, \ldots, k+1$, and pure-φ^{21}_i, $i = 1, \ldots, l+k+1$, formulae. Furthermore, every* *pure-φ_i^{12} formula having a nested G^{12}, has no P_i^{12} connectives nested in the scope of G^{12}, a pure-φ_{k+1}^{12} formula is nested in G^{12} and pure-φ_{l+1+i}^{21} formulae are nested in L^{21}, $i=0,\ldots,k$.*

Proof. By induction on the number of nested non-classical connectives in α.

Basic Case. $\alpha = C(\bar{\beta})$, where $\bar{\beta}$ is composed of boolean combinations of atoms and C is non-classical. By definition,

Each connective P_i^{12} is pure-φ_j^{12}, $i=1,\ldots,n$, $j \in \{1,\ldots,k\}$

Each connective P_i^{21} is pure-φ_j^{21}, $i=1,\ldots,m$, $j \in \{1,\ldots,l\}$

G^{12} is pure-φ_{k+1}^{12}

L^{21} is pure-φ_{l+1}^{21}.

If C is a classical connective then α is pure-φ_0.

Induction step. Assume that any formula with n nested non-classical connectives can be separated, and $\alpha = C(\bar{\beta})$, where β is composed of formulae with n nested non-classical connectives. By the induction hypothesis, $\alpha = C(\bar{\beta}')$, where $\bar{\beta}'$ is composed of boolean combinations of pure-φ_i^{12} and pure-φ_j^{21} formulae. Assume C is one of the P_i^{21}, $i=1,\ldots,m$. Exchange all pure-φ_i^{12}, $i=1,\ldots,k$, with contradictions and tautologies, as discussed in the nested cases.

All pure-φ_{l+1+i}^{21}, $i=0,\ldots,k$, cases are nested in L^{21} and, as discussed in the nested cases, can be removed outside the connective, remaining pure-φ_{l+1+i}^{21} and carrying on their special characteristics. A pure-φ_{k+1}^{12} formula is nested in $\pm G^{12}$ thus can be exchanged with a contradiction or tautology as discussed in the nested cases.

The remaining case is $\alpha = P_i^{21}(\bar{\beta})$, where each β_j is composed of pure-φ_0^{12} and pure-φ_i^{21}, $i=1,\ldots,l$. This can be separated to pure-φ_0^{12} and pure-φ_i^{21}, $i=1,\ldots,l$, formulae, since it is entirely within a grafted M_2, and any formula in P_i^{21} over the model M_2 can be separated by the assumptions to the lemma.

Assume then C is one of the P_i^{12}, $i=1,\ldots,n$. Exchange all pure-φ_i^{21} formulae, $i=1,\ldots,l$, with contradictions and tautologies, as discussed in the nested cases. Exchange all pure-φ_{l+1+i}^{21}, $i=0,\ldots,k$, cases with contradictions and tautologies as discussed in the nested cases, since they are all nested in L^{21}.

In all remaining cases there are no P_i^{12} nested within G^{12}, thus consider every G^{12} formula as an atom. The remaining case is $\alpha = P_i^{12}(\bar{\beta})$, where each β is composed of combinations of nested P_i^{12} only. This can be separated, since it is a formula over M_1, to a boolean combination of pure-φ_i^{12},

$i=0,\ldots,k$, formulae. Exchanging all the G^{12} atoms for the corresponding formulae, by definition every pure-φ_i^{12} formula remains pure-φ_i^{12}, pure-φ_{k+1}^{12} formulae may be added, and no P_i^{12} is nested within a G.

Assume C is L^{12}. By the nested cases we see that contradictions and tautologies can be exchanged for pure-φ_i^{21} formulae, $i=1,\ldots,l$. If β is a pure-φ_i^{12} formula, $i=0,\ldots,k$, then $L^{12}(\beta)$ is a pure-φ_{l+1+i}^{12} formula, respectively.

Contradictions and tautologies can be exchanged for pure-φ_i^{21} formulae, $i=l+1,\ldots,l+1+k$. If β is a pure-φ_{k+1}^{12} formula then by the nested cases the G^{21} can be eliminated resulting with a boolean combination of pure-φ_i^{21} formulae.

Assume C is G^{12}. By the nested cases we see that contradictions and tautologies can replace pure-φ_i^{12} formulae, $i=1,\ldots,k$. If β is a conjunction of pure-φ_i^{21} formulae, $i=1,\ldots,l$, then $G^{12}(\beta)$ is pure-φ_{k+1}^{12}, as seen in the nested cases. By definition, no P_i^{12} can be in the scope of G.

If β has a pure-φ_{l+1+i}^{21} formula, $i=0,\ldots,k+1$, then by the nested cases it is seen that the pure-φ_{l+1+i}^{21} formula can be taken out of G's scope, becoming a conjunct of a pure-φ_i^{12} formula.

By the generalized separation theorem, M_{12} has functional completeness. ▮

THEOREM 11. *The tense logic whose time model is $\mathscr{T}_k$ is functionally complete for all k.*

Proof. By induction on k, for $k=1,2$, we have seen in Kamp (1968) and Theorem 8. Assume we have proven it for $\mathscr{T}_k$, show for $\mathscr{T}_{k+1}$. Use the weak cover lemma.

Let M_1 be the integers with the separation to past and future and the connectives, S and U. Let M_2 be $\mathscr{T}_k$ with the connectives $U_1, S_1,\ldots, U_k, S_k, L_1,\ldots, L_{k-1}, G_1,\ldots, G_{k-1}$. Define U_{k+1} and S_{k+1} as follows:

$$\|U_{k+1}(\alpha,\beta)\|_t=1 \quad \text{iff} \quad \varphi_{k+1}(t)\wedge\exists s[\varphi_{k+1}(s)\wedge(s>t)\wedge\|\alpha\|_s=1 \wedge\forall u((s>u>t)\to\|\beta\|_u=1)]$$

$$\|S_{k+1}(\alpha,\beta)\|_t=1 \quad \text{iff} \quad \varphi_{k+1}(t)\wedge\exists s[\varphi_{k+1}(s)\wedge(t>s)\wedge\|\alpha\|_s=1 \wedge\forall u((t>u>s)\to\|\beta\|_u=1)]$$

where $\psi_1(t)=\neg\tau(t)\wedge\neg\exists x((x>t)\wedge\tau(x))$, $\psi_{k+1}(t)=\exists x\,\exists y(x\neq y)\wedge\neg(x>y)\wedge\neg(y>x)\wedge(x>t)\wedge(y>t)\wedge\psi_k(x)\wedge\psi_k(y)$, $\tau(x)=\exists z\,\exists y(z>x)\wedge(y>x)\wedge(z\neq y)\wedge\neg(z<y)\wedge\neg(y<z)\wedge\forall u(((z>u)\wedge(y>u))\to(x\geqslant u))$, $\varphi_k(t)=\psi_k(t)\wedge\neg\psi_{k+1}(t)$. Leave U_i, S_i, $i=1,\ldots,k$, unchanged:

$$\|G_k(\alpha)\|_t = 1 \quad \text{iff} \quad \varphi_{k+1}(t) \wedge \exists s[(s>t) \wedge \varphi_k(s) \wedge \neg\exists u((s>u>t) \wedge \varphi_k(u) \wedge \|\alpha\|_s = 1)]$$

$$\|L_k(\alpha)\|_t = 1 \quad \text{iff} \quad \varphi_k(t) \wedge \exists s[(t>s) \wedge \varphi_{k+1}(s) \wedge \neg\exists u((t>u>s) \wedge \varphi_{k+1}(u) \wedge \|\alpha\|_s = 1)].$$

All conditions of the weak cover lemma are met, ergo $\mathscr{T}_{k+1}$ is functionally complete. ▮

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